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## Conditional Stochastic Dominance Tests in Dynamic Settings

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### Abstract

This paper proposes nonparametric consistent tests of conditional stochastic dominance of arbitrary order in a dynamic setting. The novelty of these tests resides on the nonparametric manner of incorporating the information set into the test. The test allows for general forms of unknown serial and mutual dependence between random variables, and has an asymptotic distribution under the null hypothesis that can be easily approximated by a p-value transformation method. This method has a good finite-sample performance. These tests are applied to determine investment efficiency between US industry portfolios conditional on the performance of the market portfolio. Our analysis suggests that Utilities are the best performing sectors in normal as well as distress episodes of the market.

*JEL classification:* C1, C2, G1.

*Keywords:* Empirical processes, hypothesis testing, lower partial moments, martingale difference sequence, p-value transformation, stochastic dominance

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# 1 Introduction

During the last thirty years the interest on comparing random variables has shifted from hypothesis tests for the first and second statistical moments to more convoluted tests considering the entire distribution of the data. The reason for this is twofold. On one hand, the common belief that the underlying generating processes are nonlinear and cannot be described by simple models of mean and variance; and second, the development of sophisticated mathematical and statistical techniques based on empirical processes that allow for the comparison between distribution functions, and higher statistical moments. The interest for testing for stochastic dominance between random variables has arisen in different theoretical and applied fields within statistics, economics and recently, finance. The comparison of wealth distribution between economies has been widely investigated in the literature, see McFadden (1989), Larsen and Resnick (1993), Kaur, Prakasa Rao and Singh (1994), Anderson (1996), Davidson and Duclos (2000), Barrett and Donald (2003), among others. The close relationship between the concept of stochastic dominance and expected utility maximization for rational investors has also produced a fertile area of research in finance, see Stone (1973), Porter (1974) or Fishburn (1977). These authors also discuss the link between stochastic dominance and portfolio efficiency. More recently, Shalit and Yitzhaki (1994) and Linton, Maasoumi and Whang (2005, LMW hereafter) extend this relationship to portfolio efficiency and conditional stochastic dominance. The concept of conditional stochastic dominance has been subject to different interpretations. Thus, Shalit and Yitzhaki (1994) define marginal conditional stochastic dominance as the probabilistic conditions under which all risk averse individuals, conditional on a portfolio of assets, prefer to increase the share of a risky asset over that of another asset in the same portfolio. These authors study the implications of this definition in the efficiency of the market portfolio. LMW, on the other hand, analyze econometrically the implications of extending stochastic dominance and portfolio efficiency to a conditional, potentially dynamic, setting. These authors make allowance for serial and cross dependence between investment portfolios and develop hypothesis tests for conditional stochastic dominance with the aim of uncovering stochastically maximal investment strategies conditional on other explanatory factors. Related tests for stochastic dominance and portfolio efficiency are found in Post (2003), Kopa and Post (2009) or Scaillet and Topaloglou (2010), among others.

The statistical methods necessary to test for stochastic dominance of an arbitrary order are based on empirical processes and complex asymptotic theory. A seminal contribution is Barrett and Donald (2003), that develop tests for stochastic dominance between independent random variables in an independent and identically distributed (*iid*) framework. The asymptotic distribution of their family of test statistics is that of a Gaussian process with covariance function that depends on functions of the cumulative marginal distributions of the random variables, and hence cannot be tabulated. These authors propose a bootstrap procedure and a simulation method based on Hansen's (1996) p-value transformation to approximate the asymptotic distribution of the test. Their method also makes allowance for different sample sizes for each

random variable. The limitations of this method for the analysis of time series, as most financial applications, are obvious and led LMW to extend the method to propose consistent tests of stochastic dominance under general sampling schemes that include serial and mutual dependence between random variables. These authors work in a parametric framework in which the response variables can be linear functions of sets of explanatory variables that can contain lags of the response variables. Their method also permits to work with residuals of parametric models, and therefore, to develop tests of conditional stochastic dominance. Unfortunately, the estimation of model parameters invalidates the asymptotic theory developed in Barrett and Donald (2003) due to an extra term produced by estimation uncertainty that remains in the asymptotic distribution of the test. LMW solve this problem by implementing subsampling methods to approximate this distribution. This resampling method produces consistent estimates of the critical values of the test not only under the least favourable case given by the equality of functions but also on the boundary of the null hypothesis, see also Linton, Song and Whang (2009). The formulation of these authors is very flexible and allows for general conditioning schemes. The parametric nature of the method, potentially affected by model misspecification, and the choice of block size in the subsampling approximation of the critical values of the test are subject to criticism and discussion. A related test, in this case for marginal stochastic dominance under serial dependence, is proposed by Scaillet and Topaloglou (2010). These authors derive consistent estimates of the critical values by means of block bootstrap methods. The test statistic is computed using complex linear and mixed integer programming formulations.

The main contribution of this paper is to develop hypothesis tests of stochastic dominance of arbitrary orders under general conditioning schemes and that, unlike LMW, do not need of parametric specifications of the data generating process. By a transformation of the stochastic dominance measure in terms of lower partial moments we can apply the asymptotic theory on empirical processes for martingale difference sequences introduced in Delgado and Escanciano (2007). These tests make allowance for general forms of serial and mutual dependence between stationary processes. Also, due to their nonparametric nature, the asymptotic distribution of the tests does not suffer from estimation effects. A very appealing feature is that the asymptotic critical values can be consistently estimated by the p-value transformation method, see Hansen (1996) or van der Vaart and Wellner (1996, section on Multiplier Central Limit Theorems). The method is shown to work well for sample sizes as small as fifty observations and for high orders of stochastic dominance.

The application of these tests is to determine the efficiency of ten portfolios representing *US* industrial sectors: Nondurables, Durables, Manufactures, Energy, High Technology, Telecommunications, Shops, Health, Utilities and Others, conditional on the performance of the market portfolio. Our results show that the Utilities sector dominates stochastically the rest of sectoral portfolios for any order of stochastic dominance. The only exception is Energy that is stochastically efficient of orders one and two. This result is reinforced under market distress situations, in which all the sectors are first stochastically dominated

by Utilities. In these situations of market distress we find that Nondurables also first order stochastically dominates the rest of industrial sectors but Utilities.

The paper is structured as follows. Section 2 introduces the definition of stochastic dominance under general conditioning schemes and proposes hypothesis tests for stochastic dominance of arbitrary orders. Section 3 derives the asymptotic theory for these tests and discusses a p-value transformation method to approximate consistently the critical values of the test. In Section 4 we carry out a Monte Carlo simulation experiment to study the finite sample performance of the proposed tests. Section 5 applies this testing method to assess stochastic dominance between *US* industrial sectors conditional on market performance. Section 6 concludes; proofs and tables are gathered in an appendix.

## 2 Conditional Stochastic Dominance in Dynamic Models

This section extends the definition of stochastic dominance to general conditioning schemes and proposes consistent hypothesis tests for this condition based on nonparametric methods. Let  $(Y_t^A, X_t)_{t \in \mathbb{Z}}$  and  $(Y_t^B, X_t)_{t \in \mathbb{Z}}$  be two different  $\mathbb{R}^{1+k}$  strictly stationary multivariate time series processes, with an information set  $I_t = \{(Y_{s-1}^A, Y_{s-1}^B, X_s), t - m + 1 \leq s \leq t\}$  at time  $t$ , i.e.  $I_t \in \mathbb{R}^l$ ,  $l = (k + 2)m$ . Let  $F(y)$  be the unconditional cumulative distribution function (cdf) corresponding to  $Y_t$  and  $F_{I_t}(y) = P\{Y_t \leq y | I_t\}$  the distribution function conditional on the set  $I_t$ . The indexes  $A$  and  $B$  denote the random variables  $Y_t^A$  and  $Y_t^B$ . These random variables are defined on a compact set  $\Omega \subset \mathbb{R}$  and  $I_t$  on a compact set  $\Omega' \subset \mathbb{R}^l$  such that  $(Y_t, I_t) \in \tilde{\Omega} = \Omega \times \Omega'$ .

The definition of unconditional  $\gamma$ -stochastic dominance of  $Y_t^B$  by  $Y_t^A$  for  $\gamma \geq 1$  is

$$\Psi_\gamma^A(y) \leq \Psi_\gamma^B(y), \text{ for all } y \in \Omega \subset \mathbb{R}, \quad (1)$$

with strict inequality for some  $y$ , see Levy (2006);  $\Psi_\gamma(y) = \int_{-\infty}^y \Psi_{\gamma-1}(\tau) d\tau$  with  $\Psi_1(y) = F(y)$ . By integrating by parts  $\Psi_\gamma(y)$ , this definition can be expressed in terms of lower partial moments (*LPM*), see Stone (1973), Porter (1974) or Fishburn (1977). Condition (1) is equivalent to  $LPM_{\gamma-1}^A(y) \leq LPM_{\gamma-1}^B(y)$ , with  $LPM_\gamma(y) = \int_{-\infty}^y (y-\tau)^\gamma dF(\tau)$ , for  $\tau \leq y$ ,  $\tau, y \in \Omega \subset \mathbb{R}$ . This definition can be extended to conditional stochastic dominance. Let  $\Psi_{I_t, \gamma}(y) = \int_{-\infty}^y \Psi_{I_t, \gamma-1}(\tau) d\tau$  with  $\Psi_{I_t, 1}(y) = F_{I_t}(y)$ .

**Definition:**  $Y_t^A$   $\gamma$ -stochastic dominates  $Y_t^B$  conditional on  $I_t$ , if and only if

$$\Psi_{I_t, \gamma}^A(y) \leq \Psi_{I_t, \gamma}^B(y), \text{ for all } y \in \Omega \text{ and } t \in \mathbb{Z}. \quad (2)$$

It is simple to show that the relationships between orders of stochastic dominance in the unconditional world also hold conditionally on each  $I_t$ . Thus, first stochastic dominance implies second stochastic dom-

inance and second stochastic dominance implies third stochastic dominance and so on. Further, for first order, the concepts of conditional stochastic dominance and multivariate stochastic dominance are closely related. Multivariate stochastic dominance was studied theoretically in O'Brien and Scarsini (1991), Atkinson and Bourguignon (1982) and for applications to income distribution in McCaig and Yatchew (2007) among others. In fact, it can be also shown that for  $\gamma = 1$  conditional stochastic dominance of  $Y_t^A$  over  $Y_t^B$  given  $I_t$ , that is,  $F_{I_t}^A(y) - F_{I_t}^B(y) \leq 0$ , is a sufficient condition for the multivariate stochastic dominance of the random variable  $(Y_t^A, I_t)$  over  $(Y_t^B, I_t)$  for some  $t$  fixed. This result is immediate by noting that multivariate stochastic dominance is equivalent to  $P\{Y_t^A \leq y, I_t \leq x\} \leq P\{Y_t^B \leq y, I_t \leq x\}$ , with

$$P\{Y_t \leq y, I_t \leq x\} = E[1(Y_t \leq y)1(I_t \leq x)] = E[E[1(Y_t \leq y)|I_t]1(I_t \leq x)] = E[F_{I_t}(y)1(I_t \leq x)]. \quad (3)$$

Now, it follows that the multivariate stochastic dominance condition is equivalent to

$$E[(F_{I_t}^A(y) - F_{I_t}^B(y))1(I_t \leq x)] \leq 0.$$

Hence, conditional stochastic dominance of first order implies the multivariate counterpart. This condition also shows that multivariate stochastic dominance of first order is not sufficient to have conditional stochastic dominance since the conditional distribution of  $Y_t^A$  can be dominated by that of  $Y_t^B$  for certain  $y$  in the domain of these random variables.

Define now  $LPM_{I_t, \gamma}^Y(y) = \int_{-\infty}^y (y - \tau)^\gamma dF_{I_t}^Y$ . The characterization of conditional stochastic dominance follows analogously from the unconditional case. Thus, conditional stochastic dominance is satisfied when

$$LPM_{I_t, \gamma-1}^A(y) \leq LPM_{I_t, \gamma-1}^B(y), \text{ for all } y \in \Omega \text{ and } t \in \mathbb{Z}.$$

An alternative characterization of conditional stochastic dominance is in terms of the class of all von Neumann-Morgenstern type utility functions,  $u(y)$  with  $y \in \Omega$ , see Lemma 1 in Fishburn (1977), Shalit and Yitzhaki (p. 671, 1994) for second stochastic dominance, or Definition 2 in LMW. The extension to multivariate stochastic dominance for n-variate increasing utility functions is in Lehmann (1955) and Theorem 2 in Scarsini (1988). The following proposition extends these ideas to stochastic dominance conditional on the set  $I_t$ . For convenience, for the pair  $(Y_t, I_t)$  we shall write  $E(u, F_{I_t}) = \int_{-\infty}^{\infty} u(y) dF_{I_t}(y)$ .

**Proposition 1:**

- (i) *If A stochastically dominates B of first order, conditional on  $I_t$ , then  $E(u, F_{I_t}^A) \geq E(u, F_{I_t}^B)$  for all  $t \in \mathbb{Z}$ , and every nondecreasing real valued function  $u(y)$ , with  $y \in \Omega$ .*
- (ii) *If A stochastically dominates B of second order, conditional on  $I_t$ , then  $E(u, F_{I_t}^A) \geq E(u, F_{I_t}^B)$  for all  $t \in \mathbb{Z}$ , and every nondecreasing and concave real valued function  $u(y)$ , with  $y \in \Omega$ .*
- (iii) *If A stochastically dominates B of third order, conditional on  $I_t$ , then  $E(u, F_{I_t}^A) \geq E(u, F_{I_t}^B)$  for*

all  $t \in \mathbb{Z}$ , and every nondecreasing and concave real valued function  $u(y)$ , with  $y \in \Omega$ , for which the third derivative is negative.

In the study of portfolio choice under uncertainty risk-neutrality is characterized by increasing utility functions, risk-aversion by concave utility functions, and increasing risk-aversion by concave functions with negative third derivatives; hence the equivalence between stochastic dominance and optimal portfolio choice under uncertainty for rational investors satisfying the von Neumann-Morgenstern axioms. Similarly, Theorem 3 in Fishburn (1977) can be easily extended to show the connection between stochastic dominance and mean-risk dominance in this general conditioning scheme; a portfolio that dominates another portfolio stochastically also mean-risk dominates the portfolio conditional on  $I_t$ . Equally, if a portfolio is in the mean-risk efficient frontier it is also stochastically efficient. These results highlight the importance of developing hypothesis tests for stochastic dominance conditional on the information set  $I_t$ .

Klecan, McFadden and McFadden (1991), Anderson (1996), Davidson and Duclos (2000), and more recently, Barrett and Donald (2003) were the first to develop hypotheses for arbitrary orders of stochastic dominance in an *iid* setting. Their test, using *LPM* notation, is defined as

$$\sup_{y \in \Omega} D_{\gamma-1}(y) \leq 0, \text{ with } D_{\gamma-1}(y) = LPM_{\gamma-1}^A(y) - LPM_{\gamma-1}^B(y).$$

The stationary version of this test under the presence of serial dependence in the data is developed in Scaillet and Topaloglou (2010). LMW, on the other hand, focus on dynamic tests of conditional stochastic dominance based on the analysis of residuals of time series regression models. This residual filtering implies two problems: first, the researcher needs to propose appropriate parametrizations of  $Y_t^A$ ,  $Y_t^B$  and of their relation to the variable  $X_t$  defining the information set  $I_t$ ; and second, the test for stochastic dominance between residuals has no power against processes whose stochastic dominance is impinged by their dependence on  $I_t$ . The following example illustrates this.

**Example:** Let  $Y^A = \beta_0 + \beta_1^A X + \varepsilon^A$  and  $Y^B = \beta_0 + \beta_1^B X + \varepsilon^B$ , with  $X$  a univariate random variable,  $\varepsilon^A$  and  $\varepsilon^B$  mutually independent normal random errors,  $\beta_0 \in \mathbb{R}$  and  $0 < \beta_1^A < \beta_1^B < \infty$ . The relevant information set is  $I_t = X$ . Since  $F_X^B(y) \leq F_X^A(y)$ , for all  $y \in \Omega$  and  $x \in X$ , the random variable  $Y^B$  first stochastically dominates  $Y^A$  conditional on  $X$ . LMW propose, instead, a residual test between  $\tilde{Y}^A = \beta_0 + \varepsilon^A$  and  $\tilde{Y}^B = \beta_0 + \varepsilon^B$  obtained after filtering out the dependence on  $X$ . The null hypothesis of stochastic dominance is not rejected in either direction, since both  $\tilde{Y}^A$  and  $\tilde{Y}^B$  have the same distribution. The Monte-Carlo section also illustrates the lack of power of LMW's method against this type of alternatives.

The following family of tests considers stochastic dominance conditional on  $I_t$  in a nonparametric fashion. Let  $d_{t,\gamma-1}(y) = (y - Y_t^A)^{\gamma-1} 1(Y_t^A \leq y) - (y - Y_t^B)^{\gamma-1} 1(Y_t^B \leq y)$ , and define  $D_{I_t,\gamma-1}(y) = E[d_{t,\gamma-1}(y)|I_t]$ . It is simple to see that  $D_{I_t,\gamma-1}(y) = LPM_{I_t,\gamma-1}^A(y) - LPM_{I_t,\gamma-1}^B(y)$ . Our test of conditional stochastic

dominance is

$$H_{0,\gamma} : D_{I_t, \gamma-1}(y) \leq 0 \text{ for all } y \in \Omega \text{ and } t \in \mathbb{Z} \text{ vs. } H_{A,\gamma} : D_{I_t, \gamma-1}(y) > 0, \text{ for some } y \in \Omega \text{ or } t \in \mathbb{Z}. \quad (4)$$

By strong stationarity of  $(Y_t^A, I_t)$  and  $(Y_t^B, I_t)$ , see assumption A.1 below, it follows that  $F_{I_t}(y) = P\{Y_1 \leq y | I_1\}$ ; equally  $LP_{M_{I_t, \gamma}}(y) = LP_{M_{I_1, \gamma}}(y)$  for all  $y \in \Omega$ . The hypothesis  $H_{0,\gamma}$  can be expressed in terms of the information set  $I_1$ . Now, all the information contained in  $I_t$  for all  $t \in \mathbb{Z}$ , is reflected by all possible values in  $\Omega'$  that the random variable  $I_1$  can take. The hypothesis is

$$H_{0,\gamma} : D_{I_1, \gamma-1}(y) \leq 0 \text{ for all } (x, y) \in \tilde{\Omega} \text{ vs. } H_{A,\gamma} : D_{I_1, \gamma-1}(y) > 0, \text{ for some } (x, y) \in \tilde{\Omega}. \quad (5)$$

Under  $H_{0,\gamma}$ ,  $Y_t^A$   $\gamma$ -stochastically dominates  $Y_t^B$  conditionally on the set  $I_t$ , for all  $t \in \mathbb{Z}$ . Rejection of this hypothesis implies that  $B$  is not dominated by  $A$  for an order  $\gamma$ . If  $H_{0,\gamma}$  holds then  $H_{0,\gamma+i}$  must hold too for all  $i > 0$ . For simplicity in our analysis we will focus on the least favorable case, that abusing of notation, it will be also denoted  $H_{0,\gamma} : D_{I_1, \gamma-1}(y) = 0$ , for all  $(x, y) \in \tilde{\Omega}$ . This test can be modified to test for stochastic dominance conditional on a certain set defined by a fixed  $x \in \Omega'$ . The relevant conditional test, for  $\tilde{I}_1 = \{I_1 \leq x\}$  with  $x$  fixed, is

$$\tilde{H}_{0,\gamma} : D_{\tilde{I}_1, \gamma-1}(y) = 0, \text{ for all } y \in \Omega \text{ vs. } \tilde{H}_{A,\gamma} : D_{\tilde{I}_1, \gamma-1}(y) > 0 \text{ for some } y \in \Omega. \quad (6)$$

Barrett and Donald (2003) and particularly LMW discuss the problem of assuming equality of functions and argue that the convergence of test statistics of Kolmogorov-Smirnov and Cramér-von Mises type is not uniform over the probabilities under the null hypothesis given by the inequality condition. The latter authors solve this problem by using subsampling methods to approximate the relevant null and alternative hypotheses. This resampling method has the particular advantage of exhibiting more power for the boundary of the null hypothesis for some forms of alternative hypotheses. On the other hand, and as discussed by these authors as well, subsampling does not make use of the full sample, and as such it may lose power for alternatives that are far from the boundary. In our case, in order to use theory on empirical processes for martingale difference sequences we restrict ourselves to the least favorable case as most of existing literature on stochastic dominance hypothesis testing.

### 3 Asymptotic Theory of the Tests

The assumptions on the underlying serial dependence structure are given by the following conditions;

**A.1:**  $\{I_t, Y_t^A\}_{t \in \mathbb{Z}}$  and  $\{I_t, Y_t^B\}_{t \in \mathbb{Z}}$  are strictly stationary and ergodic processes.

**A.2:** The joint *cdfs* of  $(I_1, Y_1^A)$  and  $(I_1, Y_1^B)$  are uniformly continuous on  $\mathbb{R}^{k+1}$ . Under  $H_{0,\gamma}$ , the sequence  $d_{t,\gamma-1}(\cdot)$  is a Markov process, that is,

$$E[d_{t,\gamma-1}(\cdot)|\mathfrak{F}_t] = D_{I_t,\gamma-1}(\cdot),$$

where  $\mathfrak{F}_t = \sigma(I'_t, I'_{t-1}, \dots)$  is the  $\sigma$ -field generated by the information set up to time  $t$ .

**A.3:** The function  $d_{t,\gamma-1}(\cdot)$  is square integrable. Further,  $LPM_{I_t,2(\gamma-1)}(y) \leq C_\gamma$  with  $0 < C_\gamma < \infty$ , for all  $t \in \mathbb{Z}$ , and random variables  $Y_t^A$  and  $Y_t^B$ .

Define the empirical process

$$S_{n,\gamma-1}(x, y) = \frac{1}{\sqrt{n}} \sum_{t=1}^n \tilde{d}_{t,\gamma-1}(x, y), \quad (7)$$

with  $\tilde{d}_{t,\gamma-1}(x, y) = (d_{t,\gamma-1}(y) - D_{I_t,\gamma-1}(y)) 1(I_t \leq x)$  and covariance function

$$K_{n,\gamma-1}((x_1, y_1), (x_2, y_2)) = \frac{1}{n} \sum_{t=1}^n \tilde{d}_{t,\gamma-1}(x_1, y_1) \tilde{d}_{t,\gamma-1}(x_2, y_2). \quad (8)$$

Under these assumptions, for all  $(x, y) \in \tilde{\Omega}$ , the process  $\tilde{d}_{t,\gamma-1}(x, y)$  is a martingale difference sequence with respect to the filtration  $\mathfrak{F}_t = \sigma(I'_t, I'_{t-1}, \dots)$ , *i.e.*  $E[\tilde{d}_{t,\gamma-1}(x, y)|\mathfrak{F}_t] = 0$ . Therefore, applying a standard central limit theorem (CLT) for martingales, see Hall and Heyde (1980), the finite-dimensional distributions of  $S_{n,\gamma-1}(x, y)$  converge to those of  $S_{\infty,\gamma-1}(x, y)$ , a Gaussian process with continuous sample paths and covariance function

$$E[S_{\infty,\gamma-1}(x_1, y_1) S_{\infty,\gamma-1}(x_2, y_2)] = K_{\gamma-1}((x_1, y_1), (x_2, y_2)). \quad (9)$$

Assumption A.1 and A.2 (and A.4 below) are necessary to prove the tightness of the empirical process  $S_{n,\gamma-1}(x, y)$ . The markovian property in A.2 permits to write the conditional expectation of  $d_{t,\gamma-1}(y)$  in terms of  $D_{I_t,\gamma-1}$ . By A.3 we can use central limit theorems for martingale difference sequences.

The asymptotic distribution of empirical processes similar to (7) based on *iid* observations is widely studied in the literature, see Koul's (2002) monograph. Our interest is in testing for stochastic dominance between processes  $Y_t^A$  and  $Y_t^B$  that exhibit serial dependence of unknown form. This feature of data invalidates standard methods. To overcome this, we build on the results by Delgado and Escanciano (2007) on empirical processes for martingale difference sequences. These authors apply this theory for testing for conditional symmetry in dynamic models. Next theorem extends the finite-dimensional convergence of  $S_{n,\gamma-1}(x, y)$  to weak convergence in  $l^\infty(\tilde{\Omega})$ , the space of all uniformly bounded real functions on  $\tilde{\Omega}$ , which is equipped with the sup-norm. We use the notation  $\Rightarrow$  for weak convergence. For  $a, b \in \mathbb{R}$ , we write  $a \wedge b = \min(a, b)$ . First, we need the following condition;

**A.4:**  $K_{n,\gamma-1}((x_1, y_1), (x_2, y_2))$  converges almost surely to  $K_{\gamma-1}((x_1, y_1), (x_2, y_2))$ , uniformly over  $(x, y) \in \tilde{\Omega}$ .



**Theorem 1.** Under A.1-A.4,  $S_{n,\gamma-1}(x, y) \Rightarrow S_{\infty,\gamma-1}(x, y)$  in  $l^\infty(\tilde{\Omega})$ , with  $S_{\infty,\gamma-1}(x, y)$  a zero-mean Gaussian process with covariance function  $K_{\gamma-1}$ .

This theorem accommodates the presence of mutual dependence between  $Y_t^A$  and  $Y_t^B$ , and serial dependence of each time series. Whereas the mutual dependence conditional on  $I_t$  is explicitly considered in the asymptotic covariance function  $K_{\gamma-1}$ , the serial dependence is annihilated by the martingale difference property of  $\tilde{d}_{t,\gamma-1}(x, y)$ . This result will allow us to determine the asymptotic distribution of the different conditional stochastic dominance tests under the null hypothesis, and later, when studying the power of the tests the asymptotic distribution for local alternatives.

Under  $H_{0,\gamma}$  and assumptions A.1-A.2,  $D_{I_t,\gamma-1}(y) = 0$  for all  $t \in \mathbb{Z}$  and  $y \in \Omega$ , and  $d_{t,\gamma-1}(y)$  is a martingale difference sequence with respect to the filtration  $\{\mathfrak{F}_t\}_{t \in \mathbb{Z}}$  for each  $y \in \Omega$ . Define

$$S_{n,\gamma-1}^0(x, y) = \frac{1}{\sqrt{n}} \sum_{t=1}^n d_{t,\gamma-1}(y) 1(I_t \leq x), \quad (10)$$

with covariance function

$$K_{n,\gamma-1}^0((x_1, y_1), (x_2, y_2)) = \frac{1}{n} \sum_{t=1}^n d_{t,\gamma-1}(y_1) 1(I_t \leq x_1) d_{t,\gamma-1}(y_2) 1(I_t \leq x_2). \quad (11)$$

**Corollary 1.** Under  $H_{0,\gamma}$  and A.1-A.4, the empirical process  $S_{n,\gamma-1}^0(x, y)$  converges weakly in  $l^\infty(\tilde{\Omega})$  to a zero mean Gaussian process,  $S_{\infty,\gamma-1}^0(x, y)$ , with covariance function  $K_{\gamma-1}^0 = E[S_{\infty,\gamma-1}^0(x_1, y_1) S_{\infty,\gamma-1}^0(x_2, y_2)]$ .

This corollary allows us to test for the null hypothesis of stochastic dominance for general conditioning sets characterized by  $I_1$  and for specific conditional sets characterized by  $\tilde{I}_1 = \{I_1 \leq x\}$ , for  $x \in \Omega'$  fixed. Let  $T_{n,\gamma}^0(x)$  be a family of test statistics indexed by  $x \in \Omega'$ , defined as  $T_{n,\gamma}^0(x) = \sup_{y \in \Omega} S_{n,\gamma-1}^0(x, y)$ , and  $T_{n,\gamma}^0 = \sup_{x \in \Omega'} T_{n,\gamma}^0(x)$ . Under  $H_{0,\gamma}$  defined as the least favorable case, and applying the continuous mapping theorem (CMP), we have

$$T_{n,\gamma}^0 \xrightarrow{d} \sup_{(x,y) \in \tilde{\Omega}} S_{\infty,\gamma-1}^0(x, y). \quad (12)$$

Under a weaker version of the test given by  $\tilde{H}_{0,\gamma}$ , it follows that

$$T_{n,\gamma}^0(x) \xrightarrow{d} \sup_{y \in \Omega} S_{\infty,\gamma-1}^0(x, y), \quad (13)$$

for  $x \in \Omega'$  fixed.

Next, we show that the power of the tests  $H_{0,\gamma}$  and  $\tilde{H}_{0,\gamma}$  against a sequence of contiguous alternatives is non-trivial. Let  $D_{I_t,\gamma-1}(y) = \frac{h_{I_t,\gamma-1}(y)}{\sqrt{n}}$  with  $h_{I_t,\gamma-1}(y)$  a family of functions defined on the real line such that, for a fixed  $x \in \Omega'$ ,  $h_{n,\gamma-1}(x, \cdot) = \frac{1}{n} \sum_{t=1}^n h_{I_t,\gamma-1}(\cdot) 1(I_t \leq x) \rightarrow h_{\gamma-1}(x, \cdot) = E[h_{I_1,\gamma-1}(y) | \tilde{I}_1] P(\tilde{I}_1)$  in

$L(\Omega)$ , with  $\sup_{y \in \Omega} h_{\gamma-1}(x, \cdot) > 0$ .

**Proposition 2.** Under A.1-A.4, and  $\tilde{H}_{A,\gamma}$  defined by the set  $\tilde{I}_1 = \{I_1 \leq x\}$  with  $x \in \Omega'$  fixed,

$$S_{n,\gamma-1}^0(x, \cdot) - h_{\gamma-1}(x, \cdot) \Rightarrow S_{\infty,\gamma-1}^0(x, \cdot), \text{ in } l^\infty(\Omega). \quad (14)$$

The power of the corresponding test statistic  $T_{n,\gamma}^0(x)$  against local alternatives of this type is nontrivial since the distribution of  $S_{n,\gamma-1}^0(x, y)$  is shifted to the right for every  $x \in \Omega'$  fixed, and therefore

$$\lim_{n \rightarrow \infty} P \left( \sup_{y \in \Omega} S_{\infty,\gamma-1}^0(x, y) > T_{n,\gamma}^0(x) \right) < \alpha, \quad (15)$$

for  $x \in \Omega'$  fixed, and  $\alpha$  the significance level of the test. It is immediate to see now that if  $\tilde{H}_{0,\gamma}$  is rejected for the family of local alternatives introduced above,  $T_{n,\gamma}^0$  also has power to reject  $H_{0,\gamma}$ .

This test can be easily extended to develop tests for stochastic dominance of order  $\gamma$  of a random variable  $k^*$  over the rest of available random variables in a set, indexed by  $k = 1, 2, \dots, K$ . The hypothesis of interest is

$$\tilde{\tilde{H}}_{0,\gamma} : \max_{k \neq k^*} \sup_{(x,y) \in \bar{\Omega}} D_{I_1^{k^*k}, \gamma-1}(y) \leq 0,$$

with the information set  $I_1^{k^*k}$  defined by  $(Y_1^{k^*}, Y_1^k, X_1)$ .  $\bar{\Omega}$  is a compact set contained in the union of the supports of  $I_1^{k^*k}$  for  $k \neq k^*$ . The test statistic is  $T_{n,\gamma}^0 = \max_{k \neq k^*} \sup_{(x,y) \in \bar{\Omega}} S_{n,\gamma-1,k^*k}^0(x, y)$ , with  $S_{n,\gamma-1,k^*k}^0$  the relevant empirical process, that under the null hypothesis converges in distribution to  $\max_{k \neq k^*} \sup_{(x,y) \in \bar{\Omega}} S_{\infty,\gamma-1,k^*k}^0(x, y)$ , with  $S_{\infty,\gamma-1,k^*k}^0(x, y)$  the limiting Gaussian process corresponding to the test between  $k^*$  and  $k$ . For simplicity in the presentation the rest of results focus on bilateral hypothesis tests.

### 3.1 Approximation of the Asymptotic Critical Values

The asymptotic distribution of the process  $S_{n,\gamma-1}^0$  and hence of the test statistics  $T_{n,\gamma-1}^0(x)$  and  $T_{n,\gamma-1}^0$  is nonstandard due to the presence of nuisance parameters defining the covariance function  $K_{\gamma-1}^0$ . Asymptotic critical values for arbitrary  $\gamma$ -tests cannot be universally tabulated. Nevertheless, in this context these nuisance parameters are completely determined by the cdf of the vector  $(Y_1^A, Y_1^B, I_1)$ . Thus, knowledge of this distribution implies that the distribution of  $S_{\infty,\gamma-1}^0(x, y)$  can be approximated via Monte-Carlo simulation methods, see Koul and Ling (2006, p. 7). For  $\gamma = 1$ , for example,

$$\begin{aligned} K_0^0((x_1, y_1), (x_2, y_2)) &= E[1(Y_1^A \leq y_1 \wedge y_2)1(I_1 \leq x_1 \wedge x_2)] - E[1(Y_1^A \leq y_1)1(Y_1^B \leq y_2)1(I_1 \leq x_1 \wedge x_2)] - \\ &\quad E[1(Y_1^A \leq y_2)1(Y_1^B \leq y_1)1(I_1 \leq x_1 \wedge x_2)] + E[1(Y_1^B \leq y_1 \wedge y_2)1(I_1 \leq x_1 \wedge x_2)], \end{aligned}$$

with  $E[1(Y_1^A \leq y)1(I_1 \leq x)]$  the joint cdf of  $(Y_1^A, I_1)$  and  $E[1(Y_1^A \leq y)1(Y_1^B \leq y)1(I_1 \leq x)]$  the joint cdf of  $(Y_1^A, Y_1^B, I_1)$ . For higher orders of  $\gamma$  the asymptotic covariance function is given by higher statistical moments of the different cdfs.

In many circumstances the distribution of the vector  $(Y_1^A, Y_1^B, I_1)$  is not known. In this case there are several alternatives explored in the literature for testing for stochastic dominance, namely, simulation and *iid* bootstrap methods as in Barrett and Donald (2003), subsampling and bootstrap as in LMW, and block bootstrap for time series, as in Scaillet and Topaloglou (2010). For martingale difference sequences the naive *iid* bootstrap technique does not work, nevertheless, simulation methods derived from the p-value transformation in Hansen (1996) provide consistent estimates of the asymptotic critical values. This method simplifies enormously the computation of critical values.

Now, we operate conditionally on the sample  $\{(y_t^A, y_t^B, I_t)\}_{t=1}^n$ , and define a conditional zero mean Gaussian process  $S_{n,\gamma-1}^*(x, y)$  with covariance function  $K_{n,\gamma-1}^*((x_1, y_1), (x_2, y_2)) = \frac{1}{n} \sum_{t=1}^n d_{t,\gamma-1}(y_1) d_{t,\gamma-1}(y_2) 1(I_t \leq x_1 \wedge x_2)$ , with  $(x_1, y_1), (x_2, y_2) \in \tilde{\Omega}$ . This process can be generated from

$$S_{n,\gamma-1}^*(x, y) = \frac{1}{\sqrt{n}} \sum_{t=1}^n d_{t,\gamma-1}(y) 1(I_t \leq x) v_t, \quad (16)$$

with  $v_t$  an external standard normal distribution, and  $(x, y) \in \tilde{\Omega}$ .

**Theorem 2.** Let  $T_{n,\gamma}^*(x) = \sup_{y \in \Omega} S_{n,\gamma-1}^*(x, y)$  and  $T_{n,\gamma}^{0*} = \sup_{x \in \Omega'} T_{n,\gamma}^*(x)$ . Under  $H_{0,\gamma}$  defined by the least favorable case, and A.1-A.4,

$$T_{n,\gamma}^*(x) \xrightarrow{d} \sup_{y \in \Omega} S_{\infty,\gamma-1}^0(x, y), \quad (17)$$

for every  $x \in \Omega'$  fixed; and

$$T_{n,\gamma}^{0*} \xrightarrow{d} \sup_{(x,y) \in \tilde{\Omega}} S_{\infty,\gamma-1}^0(x, y). \quad (18)$$

Let  $p_{n,\gamma-1}^*(x) = P\left(\sup_{y \in \Omega} S_{n,\gamma-1}^*(x, y) > T_{n,\gamma}(x)\right)$  and  $p_{n,\gamma-1}^* = P\left(\sup_{(x,y) \in \tilde{\Omega}} S_{n,\gamma-1}^*(x, y) > T_{n,\gamma}^0\right)$ . Theorem 2 implies that under  $\tilde{H}_{0,\gamma}$ ,  $\lim_{n \rightarrow \infty} p_{n,\gamma-1}^*(x) = \alpha$ ; under the more general null hypothesis defined in (5) the appropriate condition is  $\lim_{n \rightarrow \infty} p_{n,\gamma-1}^*(x) \leq \alpha$ . The same argument holds for  $p_{n,\gamma-1}^*$  and  $H_{0,\gamma}$ . The asymptotic distribution of these test statistics is not directly observed but by operating conditionally on the sample, see Hansen (1996), it can be approximated to any degree of accuracy. The algorithm to compute the p-value of the test is as follows;

**Algorithm:**

1. Construct a grid of  $l_1 \times l_2$  points contained in the compact space  $A_{l_1 \times l_2} \subset \tilde{\Omega}$ , and execute the following steps for  $b = 1, \dots, B$ .

2. Generate  $\{v_t\}_{t=1}^n$  iid  $N(0, 1)$  random variables.
3. Set  $S_{n,\gamma-1}^{*(b)}(x, y) = \frac{1}{\sqrt{n}} \sum_{t=1}^n d_{t,\gamma-1}(y) 1(I_t \leq x) v_t$ , for  $(x, y) \in A_{l_1 \times l_2}$ .
4. Set  $T_{n,\gamma}^{*(b)}(x) = \sup_{y \in A_{l_2}} S_{n,\gamma-1}^{*(b)}(x, y)$ .
5. Set  $T_{n,\gamma}^{*0(b)} = \sup_{x \in A_{l_1}} T_{n,\gamma}^{*(b)}(x)$ .

For  $x \in \Omega'$  fixed and under  $H_{0,\gamma}$ , this algorithm yields a random sample of  $B$  observations from the distribution of  $T_{n,\gamma}^0(x)$  and  $T_{n,\gamma}^0$ . Using Glivenko-Cantelli and assumptions A.1-A.4, the empirical p-values conditional on the sample,

$$\hat{p}_{n,\gamma-1}^*(x) = \frac{1}{B} \sum_{b=1}^B 1(T_{n,\gamma}^{*(b)}(x) > T_{n,\gamma}^0(x)),$$

and  $\hat{p}_{n,\gamma-1}^*$ , defined analogously, converge in probability to  $p_{n,\gamma-1}^*(x)$  and  $p_{n,\gamma-1}^*$ , respectively, as  $B \rightarrow \infty$ . By Theorem 2, these measures converge to the true asymptotic p-values as  $n \rightarrow \infty$ .

## 4 Monte-Carlo Simulation Exercise

In this section we consider three different Monte Carlo simulation experiments to assess the accuracy of the subsampling of LMW and of our nonparametric method to approximate the critical values of stochastic dominance tests of orders one, two and three. The first simulation experiment studies stochastic dominance for a straightforward cross-sectional regression model with different intercept and slope parameters. The second experiment studies the test for a GARCH(1,1) structure in the second conditional moments of both random variables. Finally, experiment three shows the effect of misspecifying the data generating process on the subsampling and nonparametric stochastic dominance tests. These tests are also compared in terms of empirical power. For this, we study alternatives to the first and second exercise.

The subsampling approximation consists on dividing the original samples of  $Y_t^A$  and  $Y_t^B$  of size  $n$  in subsamples of size  $b$ , with  $b \rightarrow \infty$  and  $b/n \rightarrow 0$ , and computing the different stochastic dominance test statistics for the residuals series of each subsample. Each of the test statistics obtained from each subsample produces an observation of the empirical subsampling distribution of the test statistic that is compared against the test statistic obtained from the sample of size  $n$  to obtain the p-value. To highlight the effect of the subsampling parametric method compared to the proposed nonparametric method we assume known model parameters in exercises 1 and 2. In these cases the distortion in the reported size is entirely due to subsampling approximations and not to estimation effects. This scenario is the most unfavorable to our nonparametric proposal.

For the first experiment the data generating process is

$$Y_i^j = \alpha_0^j + \beta^j X_i + \varepsilon_i^j, \text{ with } j = A, B. \quad (19)$$

The error terms,  $\varepsilon^j$ , are assumed to be mutually independent such that all the dependence between the random variables  $Y^A$  and  $Y^B$  is through the regressor  $X$ ;  $\varepsilon^j$  are random variables that follow standardized Student-t distributions with  $\nu$  degrees of freedom. The choice of this distribution is to add flexibility to the model and to better approximate the behavior of innovations encountered in the modeling of financial time series, see Bollerslev (1987). Table 1 shows the accuracy of both subsampling and the p-value transformation method to approximate the nominal size of the test for  $\alpha_0^A = \alpha_0^B = 0$  and  $\beta^A = \beta^B = 1$ . While LMW's method underestimates the size of the test our method slightly overestimates it, especially for  $\nu = 5$ . The empirical size of the tests improves as the sample size increases but yields less accurate results for higher orders of  $\gamma$ . The presence of heavy tails in the error distribution also distorts slightly the size of the different tests.

The second experiment studies stochastic dominance for

$$Y_t^A = \alpha_0^A + a_t^A, \text{ with } a_t^A = (h_t^A)^{1/2} \varepsilon_t^A, \quad (20)$$

following a GARCH(1,1) process given by  $h_t^A = \beta_0^A + \beta_1^A (a_{t-1}^A)^2 + \beta_2^A h_{t-1}^A$ , and  $\varepsilon_t^A$  is an *iid* random variable with a Student-t distribution with  $\nu$  degrees of freedom. The process  $Y_t^B$  is defined analogously for  $\alpha_0^B$ , and  $\beta_0^B$ ,  $\beta_1^B$  and  $\beta_2^B$ . In order to be under the null hypothesis we consider as before  $\alpha_0^A = \alpha_0^B = 0$  and the same GARCH process. Table 2 shows a better approximation of the nonparametric method to the nominal size under independence between  $Y_t^A$  and  $Y_t^B$ . The results in this table for the strong cross dependence case are mixed, though. Whereas our method overestimates the size of the test, subsampling underestimates it.

For the third experiment, since the interest is in gauging misspecification effects on the tests for stochastic dominance we estimate the model parameters for computing the test statistic as well as for the subsampling stage. The data generating processes are  $Y_t^j = \alpha_0^j + \rho^j Y_{t-1}^j + Y_{t-1}^j \varepsilon_t^j$ ,  $j = A, B$ , with  $\alpha_0^A = \alpha_0^B = \rho^A = \rho^B = 0$ . We estimate, instead, an homoscedastic AR(1) process that misspecifies the conditional heteroscedasticity existing in the process. This is known to produce inefficient estimates of the model parameters and inadequate inferences. Interestingly, Table 3 shows that for the subsampling method these effects are important for first stochastic dominance (serious oversized estimates) but not so for higher orders of dominance. The nonparametric method, on the other hand, reports a rather accurate empirical size for the three orders of stochastic dominance studied. As in the previous two examples, sample size, degrees of freedom and correlation parameter have a slight distorting effect on the estimated size of the test, consistent with what theory predicts. Overall, the above simulations illustrate a similar (exercises 1 and 2) and better (exercise 3) performance of the nonparametric alternative based on empirical processes compared to the subsampling benchmark in the literature.

[INSERT TABLE 1, 2 AND 3 ABOUT HERE]

The following small exercise to analyze the empirical power of the tests under different alternative hypotheses supports these findings. As before, to highlight the effect of subsampling compared to the nonparametric method the different model parameters are assumed to be known. There is no estimation stage in the following simulations. Table 4 reports the rejection probability for three different alternative processes to model (19). These are defined by  $\alpha_0^B = 0$ ,  $\beta^B = 1.25$ ;  $\alpha_0^B = 0$ ,  $\beta^B = 1.5$  and  $\alpha_0^B = 0.25$ ,  $\beta^B = 1$ . Whereas in the first two models the stochastic dominance of A by B is impinged by a higher  $\beta$  parameter, in the third model it is due to a higher intercept that shifts the distribution of  $Y_t^B$  to the right.

This experiment shows that both subsampling and the p-value transformation are more sensitive to small variations in the intercept than in the slope parameter. More importantly, and as discussed in the example in Section 2, for first stochastic dominance the subsampling method shows no power against alternatives given by higher values of the slope regression parameter. These processes are under the null hypothesis in the framework of LMW. Our method, on the other hand, has power against these processes.

[INSERT TABLES 4 AND 5 ABOUT HERE]

The power study for the heteroscedastic case considers process (20) with  $\alpha_0^A = 0$  and  $(\beta_0^A, \beta_1^A, \beta_2^A) = (0.05, 0.10, 0.85)$ , and

$$Y_t^B = 0.1 + 0.5Y_{t-1}^B + a_t^B, \text{ with } a_t^B = (h_t^B)^{1/2}\varepsilon_t^B, \quad (21)$$

with  $h_t^B$  a GARCH(1,1) process with same parameters as  $h_t^A$ .

Since  $\alpha_0^A < \alpha_0^B$ , it is not difficult to see that  $B$  is not dominated by  $A$  stochastically, and hence the processes  $Y_t^A$  and  $Y_t^B$  are under the alternative hypothesis. The simulations confirm this and show the power of the subsampling method to reject the null hypothesis. The power in this case is only due to the difference in intercepts. The nonparametric method, on the other hand, captures differences in intercept and slope parameters and hence reports a power that is about three times as high as that of subsampling.

The good performance of our test in terms of size and power reinforce their usefulness in finite-sample applications.

## 5 Application: Stochastic Dominance Conditional on the Market Portfolio

In this section we apply our nonparametric tests of stochastic dominance to *US* sectoral portfolios conditional on the performance of the market portfolio. The data set consists of monthly excess returns on the ten equally-weighted industry portfolio obtained from the data library in Kenneth French's website, and of monthly excess returns on the market portfolio constructed as a value-weight return on all NYSE, AMEX, and NASDAQ stocks (from CRSP) minus the one-month Treasury bill rate. The period under

study is January 1960 to December 2009. Sectors are Nondurables, Durables, Manufactures, Energy, High Technology, Telecommunications, Shops, Health, Utilities and Others. We implement two different tests for each order of stochastic dominance  $\gamma = 1, 2, 3$ : a conditional test based on the performance of the market portfolio and a test conditional on market distress, that is, on the event  $\{X_t \leq 0\}$ , with  $X_t$  denoting market portfolio excess return.

Table 6 shows that the Utilities sector dominates stochastically the rest of portfolios on industrial sectors for any order of stochastic dominance and conditional on the performance of the market portfolio. The only exceptions are Energy and Telecommunications that are not dominated for first and second order, but are so from order three onwards. This result indicates that the Utilities sector is the dominant portfolio for risk-averse investors exhibiting increasing levels of risk aversion. Note that we do not need to run the reverse hypothesis test for Energy and Telecommunications because these portfolios are third order dominated by Utilities.

It is worth mentioning that this analysis is performed under the least favorable case, that is, the size of the test is less or equal to the true size of the stochastic dominance test defined by the composite inequality constraint in (5). The implications of this difference in hypotheses are that rejections of our null hypothesis imply rejections of the true stochastic dominance test; hence, the above results very strongly suggest that Energy and Telecommunications are not dominated stochastically. Unfortunately, the results suggesting the no rejection of the null hypothesis could be a bit inconclusive for p-values higher but close to the nominal size  $\alpha$ . Nevertheless, Table 6 reports large p-values, in some cases close to unity, providing clear evidence of no rejection of the null hypothesis, and hence of no significant statistical effect of considering the least favorable case rather than the test in the boundary of the null hypothesis, as stated in (5).

Under distress, defined by the occurrence of negative market portfolio returns, Utilities dominates the rest of sectors for any order of stochastic dominance. In this situation, Nondurables also performs remarkably, only dominated by Utilities. For completeness, we also show the performance of this sector against the rest of portfolios in normal periods. The results are less conclusive; Nondurables third order stochastically dominates the remaining sectors but Energy.

## 6 Concluding Remarks

In order for the concept of stochastic dominance to be fully operational it needs to be exploited dynamically. While there are many influential methods to test the hypothesis of stochastic dominance in an unconditional or marginal setting, there are just a few methods that aim to do this dynamically or conditionally on an information set. Moreover, these conditional stochastic dominance tests rely heavily on assuming an appropriate parametric structure for the dependence between the variables and hence are subject to misspecification issues.

This paper presents a nonparametric test for conditional stochastic dominance that accommodates very

easily the presence of dynamics in the variables without having to impose strong assumptions on the specific form of these dynamics. The method is, however, computationally more intensive than existing parametric methods. The asymptotic theory of the test is simple to derive and critical values can be approximated by existing p-value simulation methods. The test has good finite-sample performance and is easy to implement under a variety of conditional settings. The application to studying investment performance on sectoral indices shows that Utilities stochastically dominates most of industry sectors for arbitrary orders of stochastic dominance and all of them for  $\gamma > 2$ . Under market distress (negative market portfolio returns) Utilities dominates all the sectors for any arbitrary order.

Further research goes in the direction of extending the proposed test from the least favorable case to the boundary of the null hypothesis.



## Mathematical appendix

**Proof of Proposition 1:** The proof of this result follows similarly from the unconditional case, see Lemma 1 in Fishburn (1977).  $\square$

**Proof of Theorem 1:** For the proof of this theorem we use Theorem A.1. in Delgado and Escanciano (2007). The process  $\tilde{d}_{t,\gamma-1}(x, y) = (d_{t,\gamma-1}(y) - D_{I_t, \gamma-1}(y))1(I_t \leq x)$  is a martingale difference sequence for all  $y \in \Omega$  and  $t \in \mathbb{R}$ , with respect to the filtration  $\mathfrak{F}_t = \sigma(I'_t, I'_{t-1}, \dots)$ . This follows from assumption A.2. Further, from A.3,  $\tilde{d}_{t,\gamma-1}(y)$  is square integrable, that is,  $E[\tilde{d}_{t,\gamma-1}^2(y)|\mathfrak{F}_t] < \infty$ , for all  $y \in \Omega$ . Now, applying a standard central limit theorem (CLT) for martingales, see Hall and Heyde (1980), the finite-dimensional distributions of  $S_{n,\gamma-1}(x, y)$  converge to those of a zero mean Gaussian process with continuous sample paths and covariance function  $K_{\gamma-1}$ . To show the tightness of the process we need to prove that conditions W1 and W2 in Theorem A.1 in Delgado and Escanciano (2007) are satisfied.

(W1) By assumption A.1,  $\{Y_t^A, I_t\}_{t \in \mathbb{Z}}$  and  $\{Y_t^B, I_t\}_{t \in \mathbb{Z}}$  are strictly stationary and ergodic processes. Also, by A.4.,  $K_{n,\gamma-1}((x_1, y_1), (x_2, y_2))$  converges almost surely to  $K_{\gamma-1}((x_1, y_1), (x_2, y_2))$ , uniformly for all  $(x, y) \in \tilde{\Omega}$ . Thus, W1 is satisfied.

(W2) For every compact subset  $\tilde{\Omega}_C \subset \tilde{\Omega}$ , the family  $\tilde{d}_{t,\gamma-1}(x, y)$  is such that  $S_{n,\gamma-1}$  is a mapping from the probability space to  $l^\infty(\tilde{\Omega}_C)$  and for every  $\varepsilon > 0$  there exists a finite partition  $\mathbb{H}_\varepsilon = \{H_k; 1 \leq k \leq N_\varepsilon\}$  of  $\tilde{\Omega}_C$ , with  $N_\varepsilon$  being the elements of such partition, such that

$$\int_0^\infty \sqrt{\log(N_\varepsilon)} d\varepsilon < \infty, \quad (22)$$

and

$$\sup_{\varepsilon \in (0,1) \cap \mathbb{Q}} \frac{\alpha_{n,\gamma-1}(\mathbb{H}_\varepsilon)}{\varepsilon^2} = O_P(1), \quad (23)$$

with

$$\alpha_{n,\gamma-1}(\mathbb{H}_\varepsilon) = \max_{1 \leq k \leq N_\varepsilon} \frac{1}{n} \sum_{t=1}^n E \left[ \sup_{m_k, n_k \in H_\varepsilon} |\tilde{d}_{t,\gamma-1}(x_{mk}, y_{mk}) - \tilde{d}_{t,\gamma-1}(x_{nk}, y_{nk})|^2 | \mathfrak{F}_t \right]. \quad (24)$$

Define the semimetric  $d^2(m_k, n_k) = |D_{I_1}(y_{mk}) - D_{I_1}(y_{nk})| + |F_X(x_{mk}) - F_X(x_{nk})|$  for  $m_k = (x_{mk}, y_{mk})$  and  $n_k = (x_{nk}, y_{nk})$ . By A.2, the joint distribution functions of  $(I_1, Y_1^A)$  and  $(I_1, Y_1^B)$  are uniformly continuous on  $\mathbb{R}^{1+k}$ , and hence uniformly equicontinuous. This guarantees that for any  $\varepsilon > 0$  we can form a partition  $\mathbb{H}_\varepsilon = \{H_k; 1 \leq k \leq N_\varepsilon\}$  of  $\tilde{\Omega}$  in  $\varepsilon$ -brackets  $H_k = [m_k, n_k]$ . The set  $\{H_k\}_{k=1}^{N_\varepsilon}$  covers the compact space  $\tilde{\Omega}$ , with  $m_k \leq n_k$  and  $d^2(m_k, n_k) \leq \varepsilon^2$ . For every  $q \in \mathbb{N}$ ,  $q \geq 1$ , when  $\varepsilon = 2^{-q}$  we denote the previous partition by  $\mathbb{H}_q = \{H_{qk}; 1 \leq k \leq N_q \equiv N_{2^{-q}}\}$ . From standard results on  $VC$ -classes, see van der Vaart and Wellner (1996), condition (22) holds for these partitions.

To prove (23) we need to show the conditional quadratic variation of the empirical process  $\alpha_{n,\gamma-1}$  on the finite partition  $\mathbb{H}_q$  of  $\tilde{\Omega}_C$ .

From (24) it follows that

$$\alpha_{n,\gamma-1}(\mathbb{H}_q) \leq \quad (25)$$

$$\max_{1 \leq k \leq N_q} \left| \frac{1}{n} \sum_{t=1}^n E \left[ \sup_{m_k, n_k \in H_{qk}} (d_{t,\gamma-1}(y_{mk}) - d_{t,\gamma-1}(y_{nk}))^2 1(I_t \leq x_{mk}) | \mathfrak{S}_t \right] \right| + \quad (26)$$

$$\max_{1 \leq k \leq N_q} \left| \frac{1}{n} \sum_{t=1}^n E \left[ \sup_{m_k, n_k \in H_{qk}} d_{t,\gamma-1}^2(y_{nk}) 1(x_{mk} < I_t \leq x_{nk}) | \mathfrak{S}_t \right] \right| + \quad (27)$$

$$\max_{1 \leq k \leq N_q} \left| \frac{1}{n} \sum_{t=1}^n E \left[ \sup_{m_k, n_k \in H_{qk}} (D_{I_t, \gamma-1}(y_{mk}) - D_{I_t, \gamma-1}(y_{nk}))^2 | \mathfrak{S}_t \right] \right|. \quad (28)$$

By the definition of the semimetric  $d^2(m_k, n_k)$  for  $H_k = [m_k, n_k]$  we have  $(F_X(x_{mk}) - F_X(x_{nk}))^2 + (D_{I_t, \gamma-1}(y_{mk}) - D_{I_t, \gamma-1}(y_{nk}))^2 < \varepsilon^2$ . Now, by A.2 and given that  $I_1$  is a finite set, the marginal distribution function  $F_X$  is uniformly equicontinuous. Therefore, expression (28) is bounded in probability. Expression (27) is also bounded in probability and is proved by considering the expectation conditional on  $\mathfrak{S}_t$  and A.3 that assumes  $LPM_{I_t, 2(\gamma-1)}(y) \leq C_\gamma < \infty$ , for random variables  $Y_t^A$  and  $Y_t^B$ . Finally, expression (26) is upper bounded by

$$\max_{1 \leq k \leq N_q} \left| \frac{1}{n} \sum_{t=1}^n E \left[ ((y_{mk} - Y_t^A)^{\gamma-1} 1(Y_t^A \leq y_{mk}) - (y_{nk} - Y_t^A)^{\gamma-1} 1(Y_t^A \leq y_{nk}))^2 | \mathfrak{S}_t \right] 1(I_t \leq x_{mk}) \right| + \quad (29)$$

$$\max_{1 \leq k \leq N_q} \left| \frac{1}{n} \sum_{t=1}^n E \left[ (y_{mk} - Y_t^B)^{\gamma-1} 1(Y_t^B \leq y_{mk}) - (y_{nk} - Y_t^B)^{\gamma-1} 1(Y_t^B \leq y_{nk}))^2 | \mathfrak{S}_t \right] 1(I_t \leq x_{mk}) \right|. \quad (30)$$

Each of these expressions can be similarly studied separately. Thus, (29) is bounded by

$$\max_{1 \leq k \leq N_q} \left| \frac{1}{n} \sum_{t=1}^n \left( LPM_{I_t, 2(\gamma-1)}^A(y_{mk}) + LPM_{I_t, 2(\gamma-1)}^A(y_{nk}) \right) 1(I_t \leq x_{mk}) \right|. \quad (31)$$

By A.3,  $LPM_{I_t, 2(\gamma-1)}^A(y) \leq C_\gamma < \infty$ , for all  $t \in \mathbb{Z}$ ; and by A.1, expression (31) is bounded in probability, given  $\gamma$ . Hence W2 is satisfied and Theorem 1 is proved.

□

**Proof of Corollary 1:** Under  $H_{0,\gamma}$ ,  $D_{I_1, \gamma-1}(y) = 0$  for all  $(x, y) \in \tilde{\Omega}$ . Now, the proof immediately follows from Theorem 1. □

**Proof of Proposition 2:** Let  $D_{I_t, \gamma-1}(y) = \frac{h_{I_t, \gamma-1}(y)}{\sqrt{n}}$  with  $h_{I_t, \gamma-1}(y)$  a family of functions defined on the real line such that, for a fixed  $x \in \Omega'$ ,  $h_{n, \gamma-1}(x, \cdot) = \frac{1}{n} \sum_{t=1}^n h_{I_t, \gamma-1}(\cdot) 1(I_t \leq x) \rightarrow h_{\gamma-1}(x, \cdot) = E[h_{I_1, \gamma-1}(y) | \tilde{I}_1] P(\tilde{I}_1)$  in  $L(\Omega)$ , with  $\sup_{y \in \Omega} h_{\gamma-1}(x, \cdot) > 0$ .

From Theorem 1 it follows that

$$S_{n, \gamma-1}(x, y) = \frac{1}{\sqrt{n}} \sum_{t=1}^n d_t(y) 1(I_t \leq x) - \frac{1}{n} \sum_{t=1}^n h_{I_t, \gamma-1}(y) 1(I_t \leq x) \Rightarrow S_{\infty, \gamma-1}(x, y),$$

in  $l^\infty(\tilde{\Omega})$ . By construction,  $S_{n,\gamma-1}(x, y) = S_{n,\gamma-1}^0(x, y) - h_{n,\gamma-1}(x, y)$ , since  $\frac{1}{\sqrt{n}} \sum_{t=1}^n d_t(y) 1(I_t \leq x) = S_{n,\gamma-1}^0(x, y)$ . Now, for  $x$  fixed,  $h_{n,\gamma-1}(x, \cdot) \rightarrow h_{\gamma-1}(x, \cdot)$  in  $L(\Omega)$ , and then

$$S_{n,\gamma-1}^0(x, \cdot) - h_{\gamma-1}(x, \cdot) \Rightarrow S_{\infty,\gamma-1}^0(x, \cdot),$$

provided that

$$K_{n,\gamma-1}((x_1, y_1)(x_2, y_2)) = K_{\gamma-1}^0((x_1, y_1)(x_2, y_2)) + o_P(1),$$

uniformly on  $\Omega$ . This condition holds from assumption A.4.

□

**Proof of Theorem 2:** Define a Gaussian process  $S_{n,\gamma-1}^*$  with covariance function  $K_{n,\gamma-1}^*((x_1, y_1), (x_2, y_2)) = \frac{1}{n} \sum_{t=1}^n d_{t,\gamma-1}(y_1) 1(I_t \leq x_1) d_{t,\gamma-1}(y_2) 1(I_t \leq x_2)$ , with  $(x_1, y_1), (x_2, y_2) \in \tilde{\Omega}$ . This process can be generated from

$$S_{n,\gamma-1}^*(x, y) = \frac{1}{\sqrt{n}} \sum_{i=1}^n d_{t,\gamma-1}(y) 1(I_t \leq x) v_t, \quad (32)$$

with  $v_t$  a standard normal distribution, and  $(x, y) \in \tilde{\Omega}$ .

Let  $W$  denote the set of samples  $w = \{(y_t^A, y_t^B, I_t)\}_{t=1}^n$  for which A.1-A.4 are satisfied. Take any  $w \in W$ . Now, operate conditionally on  $w$ , so all the randomness appears in the *iid*  $N(0, 1)$  variables. Note that  $S_{n,\gamma-1}^*$  is a mean-zero Gaussian process since  $E[S_{n,\gamma-1}^*|w] = E[\frac{1}{\sqrt{n}} \sum_{i=1}^n d_{t,\gamma-1}(y) 1(I_t \leq x) v_t | w] = 0$ , conditional on  $w \in W$ . Under  $H_{0,\gamma}$ , the covariance function of this process satisfies

$$E[S_{n,\gamma-1}^*(x_1, y_1) S_{n,\gamma-1}^*(x_2, y_2) | w] = \frac{1}{n} \sum_{t=1}^n E[d_{t,\gamma-1}(y_1) d_{t,\gamma-1}(y_2) 1(I_t \leq x_1) 1(I_t \leq x_2) v_t^2 | w],$$

that for fixed  $x \in \Omega'$ , is equal to

$$\frac{1}{n} \sum_{t=1}^n d_{t,\gamma-1}(y_1) d_{t,\gamma-1}(y_2) 1(I_t \leq x) = K_{n,\gamma-1}^0((x_1, y_1)(x_2, y_2)).$$

Now, define  $T_{n,\gamma}^*(x) = \sup_{y \in \Omega} S_{n,\gamma-1}^*(x, y)$  and  $T_{n,\gamma}^{0*} = \sup_{x \in \Omega'} T_{n,\gamma}^*(x)$ . By A.4,  $K_{n,\gamma-1}^0((x_1, y_1)(x_2, y_2))$  converges almost surely to  $K_{\gamma-1}^0((x_1, y_1)(x_2, y_2))$ , uniformly on  $\tilde{\Omega}$ . Then, the finite-dimensional distributions of  $S_{n,\gamma-1}^*(x, y)$  converge to those of  $S_{\infty,\gamma-1}^0(x, y)$ . The tightness of  $S_{n,\gamma-1}^*(x, y)$ , conditional on  $w \in W$ , follows from the proof of Theorem 1 and Corollary 1. Since  $P(W) = 1$ ,  $S_{n,\gamma-1}^*(x, y) \Rightarrow S_{\infty,\gamma-1}^0(x, y)$  in  $l^\infty(\tilde{\Omega})$ . By the continuous mapping theorem applied to the supremum functional the results in the theorem hold. □

## References

- [1] Anderson, G.J. (1996). “Nonparametric Tests of Stochastic Dominance in Income Distributions.” *Econometrica* 64, 1183-1193.
- [2] Atkinson, A.B., and F. Bourguignon (1982). “The Comparison of Multi-Dimensioned Distributions of Economic Status.” *Review of Economic Studies* 49, 183-201.
- [3] Barrett, G. and S. Donald (2003). “Consistent Tests for Stochastic Dominance.” *Econometrica* 71, 71-104.
- [4] Bollerslev, T. (1987). “A Conditionally Heteroskedastic Time Series Model for Speculative Prices and Rates of Return.” *The Review of Economics and Statistics* 69, 3, 542-547.
- [5] Davidson, R., and J.Y. Duclos (2000). “Statistical Inference for Stochastic Dominance and for the Measurement of Poverty and Inequality.” *Econometrica* 68, 1435-1464.
- [6] Delgado, M.A., and J.C. Escanciano (2007). “Nonparametric Tests for Conditional Symmetry in Dynamic Models.” *Journal of Econometrics* 141, 652-682.
- [7] Fishburn, P.C. (1977). “Mean-Risk Analysis with Risk Associated with Below-Target Returns.” *The American Economic Review* 67, 2, 116-126.
- [8] Hall, P., and C.C. Heyde (1980). *Martingale Limit Theory and its Applications*. Wiley, New York.
- [9] Hansen, B.E. (1996). “Inference when a Nuisance Parameter is not Identified Under the Null Hypothesis.” *Econometrica* 64, 2, 413-430.
- [10] Kaur, A., B. L. Prakasa Rao, and H. Singh (1994). “Testing for Second-Order Stochastic Dominance of Two Distributions.” *Econometric Theory*, 10, 849-866.
- [11] Klecan, L., R. McFadden and R. McFadden (1991). “A Robust Test for Stochastic Dominance.” Working Paper, Department of Economics, MIT.
- [12] Kopa, M., and T. Post (2009). “A Portfolio Optimality Test based on the First-Order Stochastic Dominance Criterion.” *Journal of Financial and Quantitative Analysis* 44, 1103-1124.
- [13] Koul, L.H. (2002). *Weighted Empirical Processes in Dynamic Nonlinear Models*. Lecture Notes in Statistics, vol. 66, Springer, Berlin, 2<sup>nd</sup> Edition.
- [14] Koul, L.H., and S. Ling (2006). “Fitting an error distribution in some heterocedastic time series models.” *The Annals of Statistics* 34, 994-1012.

- [15] Larsen, G.A., and B.G. Resnick (1993). "Bootstrapping a Distance Test for Stochastic Dominance Analysis." *Review of Quantitative Finance and Accounting* 3 (1), 61-69.
- [16] Lehmann, E. (1955). "Ordered Families of Distributions." *Annals of Mathematical Statistics* 16, 399-419.
- [17] Levy, H. (2006). *Stochastic Dominance: Investment Decision Making under Uncertainty*. Springer, New York, 2<sup>nd</sup> Edition.
- [18] Linton, O., E. Maasoumi, and Y.J. Whang (2005). "Consistent Testing for Stochastic Dominance under General Sampling Schemes." *Review of Economic Studies* 72, 735-765.
- [19] Linton, O., K. Song, and Y.J. Whang (2009). "Bootstrap Tests of Stochastic Dominance with Asymptotic Similarity on the Boundary." LSE STICERD Research Paper No. EM527.
- [20] McCaig, B., and A. Yatchew (2007). "International Welfare Comparisons and nonparametric Testing of Multivariate Stochastic Dominance." *Journal of Applied Econometrics* 22, 951-969.
- [21] McFadden, D. (1989). "Testing for Stochastic Dominance." In *Studies in the Economics of Uncertainty*, edited by T.B. Fomby and T.K. Seo. New York: Springer-Verlag.
- [22] O'Brien, G.L., and M. Scarsini (1991). "Multivariate Stochastic Dominance and Moments." *Mathematics of Operations Research* 16, 2, 382-389.
- [23] Porter, R.B. (1974) "Semivariance and Stochastic Dominance: A Comparison." *American Economic Review* 64, 200-204.
- [24] Post, T. (2003). "Empirical Tests for Stochastic Dominance Efficiency." *Journal of Finance* 58 (5), 1905-1932.
- [25] Scaillet, O., and N. Topaloglou (2010). "Testing for Stochastic Dominance Efficiency." *Journal of Business and Economic Statistics* 28 (1), 169-180.
- [26] Scarsini, M. (1988). "Dominance Conditions for Multivariate Utility Functions." *Management Science* 34, 4, 454-460.
- [27] Shalit, H., and S. Yitzhaki (1994). "Marginal Conditional Stochastic Dominance." *Management Science* 40, 5, 670-684.
- [28] Stone, B.K. (1973). "A General Class of Three-Parameter Risk Measures." *Journal of Finance* 28, 675-685.
- [29] van der Vaart, A.W., and J.A. Wellner (1996). *Weak Convergence and Empirical Processes*. New York, Springer.

SIZE STUDY:

Table 1.  $Y_t^j = \alpha_0^j + \beta^j X_t + \varepsilon_t^j$ ,  $i = 1, \dots, n$ ,  $\varepsilon_t^j \sim t_\nu$ , with  $j = A, B$  and  $\nu = 30, 5$ .  $\alpha_0^A = \alpha_0^B = 0$  and  $\beta^A = \beta^B = 1$ .  $LMW$  denotes the subsampling test of LMW and  $p$ -value the method introduced in the paper.

	$\gamma = 1$			$\gamma = 2$			$\gamma = 3$		
	$n = 50$		$n = 500$	$n = 50$		$n = 500$	$n = 50$		$n = 500$
Nominal size	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050	0.050
$\nu = 30$									
p-value	0.110	0.065	0.105	0.055	0.110	0.065	0.085	0.035	0.095
$LMW_{sub}$	0.100	0.050	0.085	0.045	0.075	0.040	0.080	0.040	0.035
$\nu = 5$									
p-value	0.150	0.060	0.125	0.050	0.135	0.095	0.140	0.075	0.060
$LMW_{sub}$	0.100	0.060	0.045	0.020	0.070	0.020	0.050	0.035	0.015

Table 2.  $Y_t^j = \alpha_0^j + a_t^j$ ,  $a_t^j = (h_t^{1/2})^j \varepsilon_t^j$ ,  $h_t^j = 0.05 + 0.10(a_{t-1}^j)^2 + 0.85h_{t-1}^j$ ,  $\varepsilon_t^j \sim t_\nu$ , with  $j = A, B$ ,  $\nu = 30, 5$  and  $\rho = corr(\varepsilon^A, \varepsilon^B)$ .  $\alpha_0^A = \alpha_0^B = 0$ .

	$n = 50$			$n = 500$			$n = 50$			$n = 500$		
	$n = 50$		$n = 500$	$n = 50$		$n = 500$	$n = 50$		$n = 500$	$n = 50$		$n = 500$
Nominal size	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050
$\nu = 30, \rho = 0$												
p-value	0.100	0.035	0.110	0.030	0.070	0.010	0.090	0.060	0.080	0.020	0.130	0.040
$LMW_{sub}$	0.145	0.115	0.075	0.040	0.095	0.035	0.070	0.050	0.060	0.020	0.075	0.045
$\nu = 30, \rho = 0.8$												
p-value	0.180	0.070	0.110	0.070	0.240	0.100	0.160	0.100	0.210	0.140	0.120	0.090
$LMW_{sub}$	0.145	0.075	0.065	0.035	0.075	0.045	0.060	0.035	0.075	0.035	0.045	0.035
$\nu = 5, \rho = 0$												
p-value	0.090	0.040	0.100	0.020	0.100	0.040	0.110	0.030	0.110	0.040	0.120	0.070
$LMW_{sub}$	0.145	0.095	0.095	0.035	0.070	0.035	0.045	0.020	0.075	0.045	0.025	0.015
$\nu = 5, \rho = 0.8$												
p-value	0.190	0.120	0.120	0.070	0.200	0.120	0.160	0.090	0.190	0.130	0.140	0.090
$LMW_{sub}$	0.090	0.055	0.065	0.035	0.070	0.040	0.050	0.015	0.055	0.030	0.035	0.015

Table 3. True DGP:  $Y_t^j = \alpha_0^j + \rho^j Y_{t-1}^j + Y_{t-1}^j \varepsilon_t^j$ ,  $\varepsilon_t^j \sim t_\nu$ , with  $j = A, B$  and  $\nu = 30, 5$ .  $\alpha_0^A = \alpha_0^B = \rho^A = \rho^B = 0$ .

	$n = 50$			$n = 500$			$n = 50$			$n = 500$		
	$n = 50$		$n = 500$	$n = 50$		$n = 500$	$n = 50$		$n = 500$	$n = 50$		$n = 500$
Nominal size	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050
$\nu = 30, \rho = 0$												
p-value	0.110	0.060	0.140	0.070	0.100	0.030	0.120	0.070	0.130	0.050	0.120	0.070
$LMW_{sub}$	0.140	0.070	0.250	0.185	0.135	0.075	0.140	0.085	0.110	0.070	0.095	0.035
$\nu = 30, \rho = 0.8$												
p-value	0.130	0.070	0.070	0.020	0.190	0.130	0.110	0.080	0.230	0.150	0.140	0.070
$LMW_{sub}$	0.145	0.070	0.190	0.135	0.095	0.040	0.125	0.095	0.090	0.030	0.115	0.065
$\nu = 5, \rho = 0$												
p-value	0.100	0.050	0.110	0.040	0.110	0.060	0.120	0.080	0.130	0.050	0.110	0.110
$LMW_{sub}$	0.165	0.085	0.200	0.150	0.115	0.050	0.085	0.060	0.105	0.040	0.110	0.075
$\nu = 5, \rho = 0.8$												
p-value	0.120	0.080	0.070	0.030	0.180	0.110	0.100	0.080	0.190	0.110	0.140	0.080
$LMW_{sub}$	0.165	0.100	0.220	0.190	0.160	0.070	0.140	0.100	0.105	0.040	0.095	0.065

POWER STUDY:

Table 4.  $Y_t^j = \alpha_0^j + \beta^j X_t + \varepsilon_t^j$ ,  $i = 1, \dots, n$ ,  $\varepsilon_t^j \sim t_\nu$ , with  $j = A, B$ ,  $\nu = 30, 5$ , and  $\gamma = 1$ .

	$\alpha_0^A = \alpha_0^B = 0, \beta^A = 1, \beta^B = 1.25$ $n = 500$										$\alpha_0^A = \alpha_0^B = 0, \beta^A = 1, \beta^B = 1.50$ $n = 500$				$\alpha_0^A = 0, \beta^A = 1, \alpha_0^B = 0.25 \beta^B = 1$ $n = 500$			
Nominal size	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050		
$\nu = 30$																		
p-value	0.125	0.080	0.200	0.085	0.210	0.120	0.775	0.575	0.320	0.210	0.975	0.940						
$LMW_{sub}$	0.100	0.050	0.085	0.045	0.100	0.050	0.085	0.045	0.275	0.145	0.895	0.810						
$\nu = 5$																		
p-value	0.200	0.095	0.280	0.095	0.300	0.180	0.760	0.660	0.435	0.275	0.975	0.970						
$LMW_{sub}$	0.100	0.060	0.045	0.020	0.100	0.060	0.045	0.020	0.320	0.205	0.940	0.875						

Table 5.  $Y_t^A = a_t^j$  and  $Y_t^B = 0.1 + 0.5Y_{t-1}^B + a_t^j$ ,  $a_t^j = h_t^{1/2} \varepsilon_t^j$ ,  $h_t^j = 0.05 + 0.10(a_{t-1}^j)^2 + 0.85h_{t-1}^j$ ,  $\varepsilon_t^j \sim t_\nu$ , with  $j = A, B$ ,  $\nu = 30, 5$  and  $\rho = corr(\varepsilon^A, \varepsilon^B)$ .

	$n = 50$		$\gamma = 1$		$n = 500$		$n = 50$		$\gamma = 2$		$n = 500$		$n = 50$		$\gamma = 3$		$n = 500$	
Nominal size	0.100	0.050	0.050	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050	0.100	0.050	
$\nu = 30, \rho = 0$																		
p-value	0.420	0.330	0.770	0.700	0.400	0.290	0.690	0.630	0.330	0.250	0.520	0.430						
$LMW_{sub}$	0.185	0.125	0.375	0.280	0.135	0.080	0.440	0.300	0.115	0.060	0.395	0.255						
$\nu = 30, \rho = 0.8$																		
p-value	0.600	0.500	0.990	0.990	0.550	0.500	0.980	0.970	0.500	0.410	0.760	0.710						
$LMW_{sub}$	0.260	0.180	0.635	0.495	0.205	0.140	0.885	0.800	0.190	0.100	0.850	0.765						
$\nu = 5, \rho = 0$																		
p-value	0.430	0.320	0.880	0.800	0.460	0.360	0.700	0.640	0.370	0.300	0.470	0.370						
$LMW_{sub}$	0.155	0.115	0.405	0.260	0.145	0.075	0.345	0.240	0.130	0.090	0.315	0.230						
$\nu = 5, \rho = 0.8$																		
p-value	0.600	0.530	0.990	0.980	0.600	0.530	0.960	0.950	0.460	0.410	0.720	0.680						
$LMW_{sub}$	0.250	0.175	0.720	0.585	0.215	0.155	0.900	0.815	0.230	0.1500	0.820	0.730						

Table 6. Empirical p-values for tests of stochastic dominance of order  $\gamma = 1, 2, 3$  for industry portfolio.

Utilities against		$H_{0,\gamma}$			$\tilde{H}_{0,\gamma}$ for $\{X_t \leq 0\}$		
$\gamma =$		First	Second	Third	First	Second	Third
Nondurables		0.930	0.770	1.000	1.000	1.000	1.000
Durables		0.565	0.385	0.975	1.000	1.000	1.000
Manufactures		0.600	0.460	1.000	1.000	1.000	1.000
Energy		0.000	0.000	0.390	1.000	1.000	1.000
High Technology		0.130	0.090	0.995	1.000	1.000	1.000
Telecom		0.030	0.020	0.990	1.000	1.000	1.000
Shops		0.355	0.255	0.985	1.000	1.000	1.000
Health		0.120	0.080	0.965	1.000	1.000	1.000
Others		0.630	0.505	1.000	1.000	1.000	1.000

  

Nondurables against		$H_{0,\gamma}$			$\tilde{H}_{0,\gamma}$ for $\{X_t \leq 0\}$		
$\gamma =$		First	Second	Third	First	Second	Third
Durables		0.530	0.380	0.460	1.000	1.000	1.000
Manufactures		0.635	0.600	0.980	1.000	1.000	1.000
Energy		0.000	0.000	0.000	0.990	1.000	0.990
High Technology		0.050	0.035	0.540	1.000	1.000	1.000
Telecom		0.000	0.010	0.255	1.000	1.000	1.000
Shops		0.130	0.270	0.690	1.000	1.000	1.000
Health		0.070	0.060	0.305	1.000	1.000	1.000
Utilities		0.000	0.000	0.000	0.000	0.000	0.000
Others		0.060	0.060	0.475	0.490	0.435	0.510